OF RAINBOWS AND RINGS:
The mathematics of the Stokes phenomenon

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Universität Bern

Overview

1. Rainbows
   - Descartes
   - Airy
   - Stokes

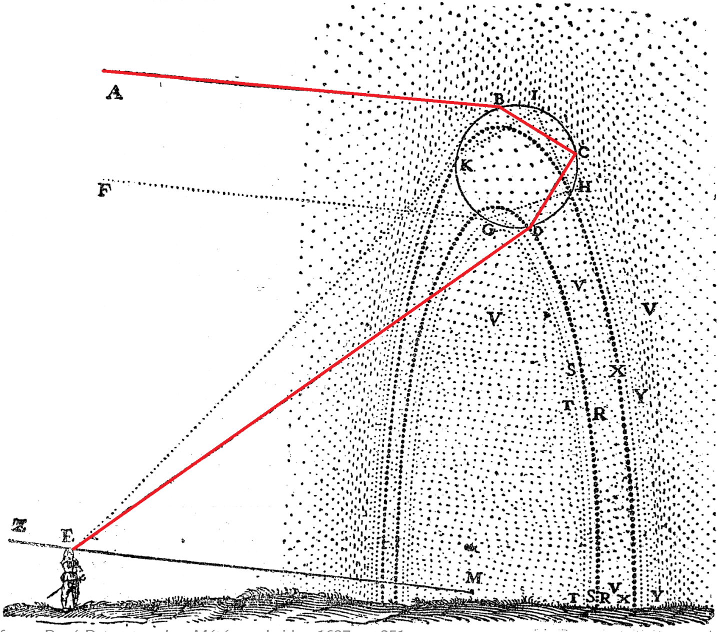
2. Rings and differential equations
   - Riemann
   - Hilbert
   - Noether
   - Deligne
   - Kashiwara

Rainbows

What is a rainbow?

Descartes: De l'arc-en-ciel

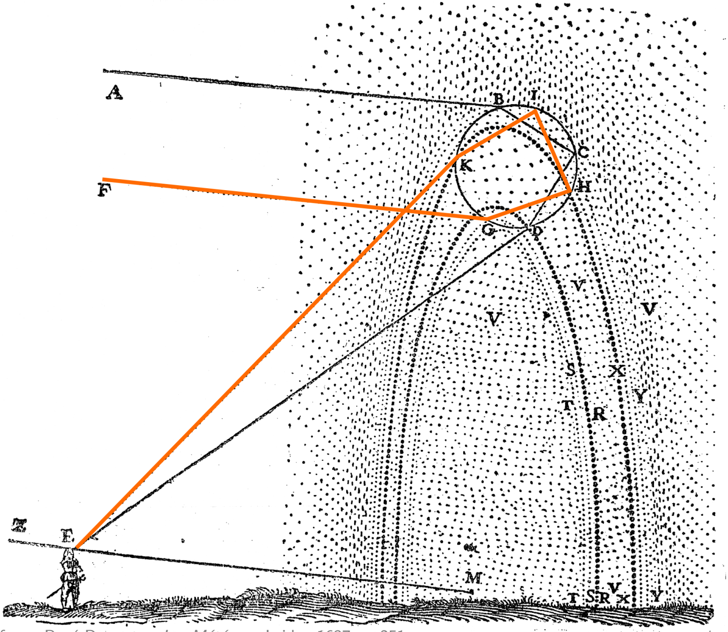
René Descartes
(1596–1650)



from: René Descartes, *Les Météores*, Leiden 1637, p. 251

What is a rainbow?

Descartes: De l'arc-en-ciel



from: René Descartes, *Les Météores*, Leiden 1637, p. 251

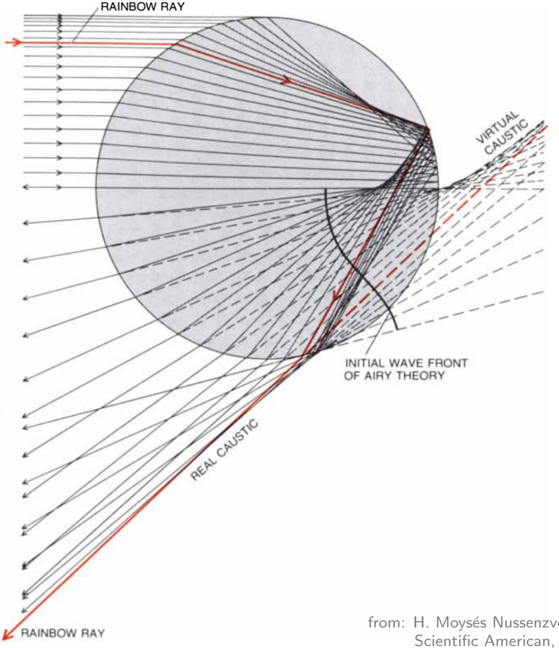
Supernumerary rainbows

By Eric Rolph at English Wikipedia

Supernumerary rainbows

from: Smithsonian Magazine, November 2010

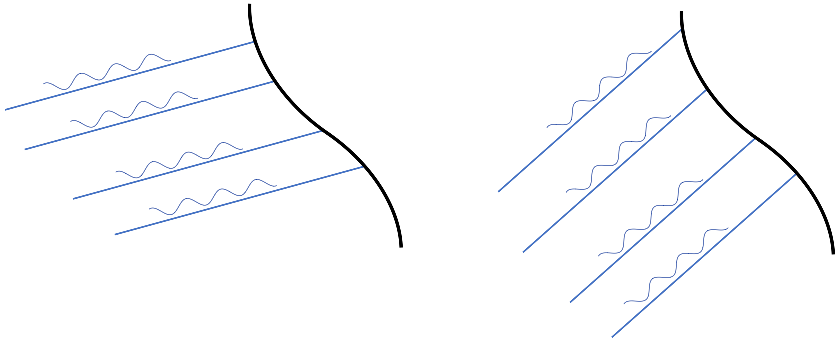
Supernumerary rainbows



from: H. Moysés Nussenzveig, *The Theory of the Rainbow*, Scientific American, vol. 236 (1977), p. 121

The Airy function

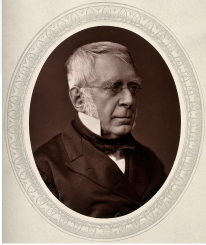
For the observer, the situation is as follows:

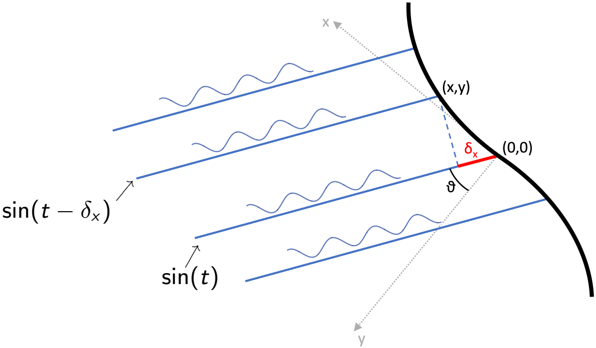


⇝ The light one sees is the superposition of light waves emerging from all the points of the wavefront. These can have different phases, leading to interference, and the latter depends on the observer's angle.

The Airy function

Sir George Biddell Airy
(1801–1892)





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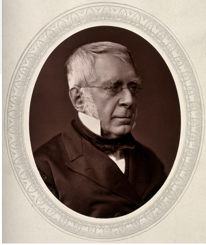

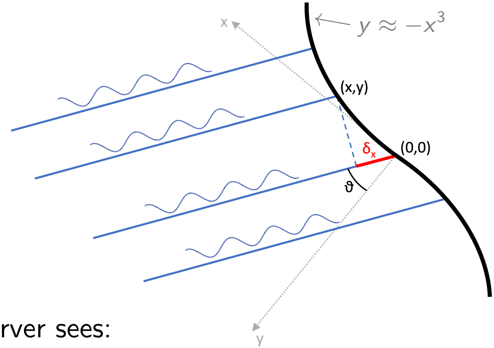

$$\begin{aligned}\delta_x &= x \sin\vartheta - y\cos\vartheta \\ &\approx x\vartheta - y \\ &= x\vartheta + x^3\end{aligned}$$

What the observer sees:

$$\int_{-\infty}^{\infty} \sin(t-\delta_x)\mathrm{d}x = \int_{-\infty}^{\infty} \sin(t)\cos(\delta_x)\mathrm{d}x + \cancel{\int_{-\infty}^{\infty} \cos(t)\sin(\delta_x)\mathrm{d}x}$$

$$= \sin(t)\cdot 2\int_0^{\infty}\cos(\delta_x)\mathrm{d}x = \ldots$$

This is a light wave with amplitude

$$\mathrm{Ai}(\vartheta) = \int_0^{\infty} \cos\frac{\pi}{2}(x^3 + x\vartheta)\mathrm{d}x$$

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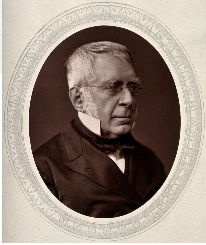
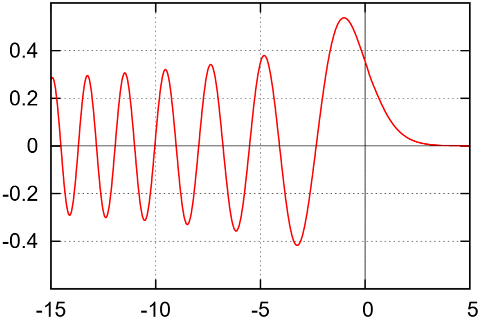




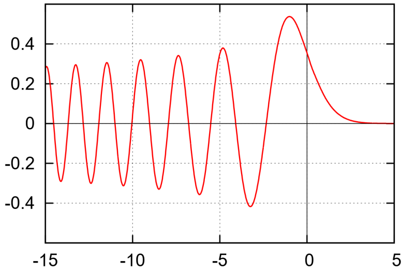
$$\mathrm{Ai}(\vartheta) = \int_0^\infty \cos \frac{\pi}{2}(x^3 + x\vartheta)\mathrm{d}x$$

Roots of the Airy function: Airy's computations

Values of m	Corresponding values of $\int_w \cos \frac{\pi}{2}(w^3 - m \,.\, w)$ from 0 to $\frac{1}{0}$	Squares of the last Numbers	Values of m	Corresponding values of $\int_w \cos \frac{\pi}{2}(w^3 - m \,.\, w)$ from 0 to $\frac{1}{0}$	Squares of the last Numbers
−4·0	+0·00298	0·0000089	+0·6	+0·91431	0·83597
−3·8	+ ·00431	·0000186	+0·8	+0·97012	0·94114
−3·6	+ ·00618	·0000382	+1·0	+1·00041	1·00082
−3·4	+ ·00879	·0000773	+1·2	+0·99786	0·99572
−3·2	+ ·01239	·0001536	+1·4	+ ·95606	·91406
−3·0	+ ·01730	·000299	+1·6	+ ·87048	·75773
−2·8	+ ·02393	·000573	+1·8	+ ·73939	·54670
−2·6	+ ·03277	·001074	+2·0	+ ·56490	·31912
−2·4	+ ·04442	·00197	+2·2	+ ·35366	·12508
−2·2	+ ·05959	·00355	+2·4	+ ·11722	·013741
−2·0	+ ·07908	·00625	+2·6	− ·12815	·016422
−1·8	+ ·10377	·01077	+2·8	− ·36237	·13131
−1·6	+ ·13461	·01812	+3·0	− ·56322	·31721
−1·4	+ ·17254	·02977	+3·2	− ·70874	·50231
−1·2	+ ·21839	·04769	+3·4	− ·78018	·60868
−1·0	+ ·27283	·07444	+3·6	− ·76516	·58547
−0·8	+ ·33621	·11304	+3·8	− ·66054	·43631
−0·6	+ ·40839	·16678	+4·0	−0·47446	0·22511
−0·4	+ ·48856	·23869			
−0·2	+ ·57507	·33071			
0·0	+ ·66527	·44259			
+0·2	+ ·75537	·57059			
+0·4	+0·84040	0·70628			

from: George B. Airy, *On the Intensity of Light in the neighbourhood of a Caustic*, Transactions of the Cambridge Philosophical Society, vol. VI (1837), p. 390

Roots of the Airy function: Airy's computations



Values of m	Corresponding values of $\int_w \cos\frac{\pi}{2}(w^3 - m\,.\,w)$ from 0 to $\frac{1}{0}$	Squares of the last Numbers
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−1·0	+ ·27283	·07444
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−0·4	+ ·48856	·23869
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+0·6	+0·91431	0·83597
+0·8	+0·97012	0·94114
+1·0	+1·00041	1·00082
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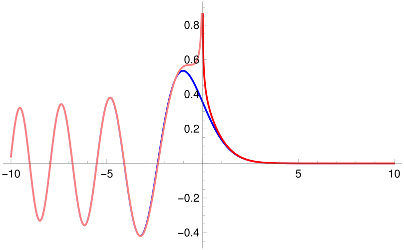
Stokes's approach

Sir George Gabriel Stokes
(1819–1903)

Idea: Approximate $\mathrm{Ai}(x)$ for **large** absolute values of x

for $x \to +\infty$: $\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}}$

for $x \to -\infty$: $\mathrm{Ai}(x) \sim \frac{1}{\sqrt{\pi}} |x|^{-\frac{1}{4}} \sin\left(\frac{2}{3}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right)$



i	m	diff.	i	m	diff.
1	2·4955	1·8676	26	26·1602	·6730
2	4·3631	1·5291	27	26·8332	·6647
3	5·8922	1·3514	28	27·4979	·6567
4	7·2436	1·2352	29	28·1546	·6491
5	8·4788	1·1512	30	28·8037	·6419
6	9·6300	1·0861	31	29·4456	·6349
7	10·7161	1·0335	32	30·0805	·6284
8	11·7496	·9899	33	30·7089	·6219
9	12·7395	·9529	34	31·3308	·6159
10	13·6924	·9208	35	31·9467	·6100
11	14·6132	·8927	36	32·5567	·6043
12	15·5059	·8676	37	33·1610	·5989
13	16·3735	·8452	38	33·7599	·5936
14	17·2187	·8250	39	34·3535	·5885
15	18·0437	·8065	40	34·9420	·5836
16	18·8502	·7897	41	35·5256	·5788
17	19·6399	·7740	42	36·1044	·5742
18	20·4139	·7597	43	36·6786	·5698
19	21·1736	·7463	44	37·2484	·5655
20	21·9199	·7337	45	37·8139	·5612
21	22·6536	·7221	46	38·3751	·5572
22	23·3757	·7111	47	38·9323	·5532
23	24·0868	·7008	48	39·4855	·5494
24	24·7876	·6909	49	40·0349	·5456
25	25·4785	·6817	50	40·5805	

from: George G. Stokes, *On the Numerical Calculation of a class of Definite Integrals and Infinite Series*, Transactions of the Cambridge Philosophical Society, vol. IX (1850), p. 349

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I have calculated the first 15 vanishing points from the formula (B). The calculation did not take $1\frac{1}{2}$ hour.

from: Correspondence with G. B. Airy, *May 12th*, 1848

i	m	diff.	i	m	diff.
1	2·4955	1·8676	26	26·1602	·6730
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The Stokes phenomenon

for $x \to +\infty$: $\quad \mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3} x^{\frac{3}{2}}}$

for $x \to -\infty$: $\quad \mathrm{Ai}(x) \sim \frac{1}{\sqrt{\pi}} |x|^{-\frac{1}{4}} \sin\left(\frac{2}{3}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right)$

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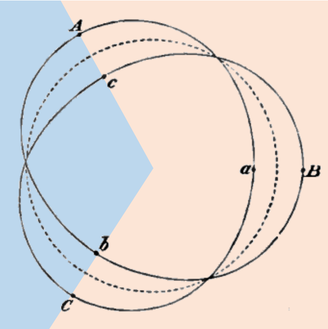
for $x \to -\infty$: $\quad \mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3} x^{\frac{3}{2}}} + i \cdot \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{\frac{2}{3} x^{\frac{3}{2}}}$

The Airy function in a complex variable:

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{\frac{3}{2}}} + i \cdot \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3} z^{\frac{3}{2}}}$$



$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{\frac{3}{2}}}$$

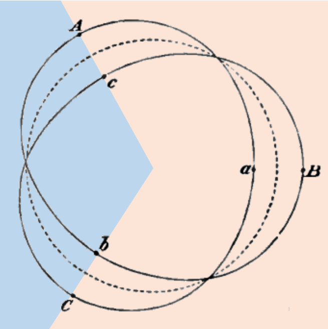
The Stokes phenomenon

As θ passes through the critical value, the inferior term enters as it were into a mist, is hidden for a little from view, and comes out with its coefficient changed.

from: George G. Stokes, *On the discontinuity of arbitrary constants that appear as multipliers of semi-convergent series. (A letter to the Editor.)*, Acta Math., vol. 26 (1902), p. 396

The Airy function in a complex variable:

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{\frac{3}{2}}} + i \cdot \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3} z^{\frac{3}{2}}}$$



$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{\frac{3}{2}}}$$

Rings and differential equations

Monodromy

Bernhard Riemann
(1826–1866)

Differential equations in the complex variable z:

- $z \cdot f'(z) + f(z) = 0 \quad \rightsquigarrow f(z) = c \cdot \frac{1}{z} \quad (c \in \mathbb{C})$

  1-dimensional complex vector space of solutions on $\mathbb{C} \setminus \{0\}$

- $2z \cdot f'(z) + f(z) = 0 \quad \rightsquigarrow f(z) = c \cdot \frac{1}{\sqrt{z}} \quad (c \in \mathbb{C})$

  locally 1-dimensional complex vector space of solutions,
  monodromy = multiplication by -1

- $z \cdot f''(z) + f'(z) = 0 \quad \rightsquigarrow f_1(z) = c_1 \cdot 1, \quad f_2(z) = c_2 \cdot \log(z)$

  locally 2-dimensional complex vector space of solutions,
  monodromy matrix $\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$

Hilbert's 21st problem

David Hilbert
(1862–1943)

21. Beweis der Existenz linearer Differentialgleichungen mit vorgeschriebener Monodromiegruppe.

Aus der Theorie der linearen Differentialgleichungen mit einer unabhängigen Veränderlichen z möchte ich auf ein wichtiges Problem binweisen, welches wohl bereits Riemann im Sinne gehabt hat, und welches darin besteht, zu zeigen, daß es stets *eine lineare Differentialgleichung der Fuchsschen Klasse mit gegebenen singulären Stellen und einer gegebenen Monodromiegruppe giebt.*

from: D. Hilbert, *Mathematische Probleme*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1900), pp. 289–290

21. Proof of the Existence of Linear Differential Equations Having a Prescribed Monodromic Group.

In the theory of linear differential equations with one independent variable z, I wish to indicate an important problem, one which very likely Riemann himself may have had in mind. This problem is as follows: *To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromic group.*

from: D. Hilbert, *Mathematical Problems*, Bull. Amer. Math. Soc. 8 (1902), pp. 470–471

Hilbert's 21st problem

Question (Hilbert)

Is the map

$$\{\textit{regular linear DEs}\} \longrightarrow \{\textit{monodromy data}\}$$

surjective?

Question

Is the map injective? Is it a 1:1-correspondence?

We need an algebraic/a categorical approach to this problem!

Differential operators

Linear differential operators with polynomial coefficients:

$$P = \sum_{j,k=0}^{n} a_{jk} z^j \frac{\mathrm{d}^k}{\mathrm{d}z^k} = \sum_{j,k=0}^{n} a_{jk} z^j \partial^k, \qquad a_{jk} \in \mathbb{C}$$

The set of these operators is called the **Weyl algebra**, denoted by $D = \mathbb{C}[z]\langle\partial\rangle$.

D has two natural operations

- addition
- composition of operators
  e.g. $z \circ \partial = z\partial$
  $\partial \circ z = z\partial + 1$

$\Rightarrow$ D is a (non-commutative) ring.

D-modules

Emmy Noether
(1882–1935)

D-module = a (left) module over the ring D

Example: Instead of the differential equation $(z\partial + 1)f(z) = 0$, we consider the D-module

$$M = D/(z\partial + 1)$$

where $(z\partial + 1)$ is the (left) ideal in D generated by $z\partial + 1$.

How to speak about solutions of the differential equation?

$$\mathrm{Hom}_D(M, \mathcal{O}) \cong \{f \in \mathcal{O} \mid (x\partial + 1)f = 0\}$$

(Here, $\mathcal{O}$ is a set of functions, e.g. holomorphic functions outside 0.)

Riemann–Hilbert correspondences

Pierre Deligne
(*1944)

Let X be an algebraic variety.

Theorem (P. Deligne 1970)

There is an equivalence of categories

$$\{\textit{regular integrable connections on } X\} \xrightarrow{\sim} \{\textit{local systems on } X\}$$

↑ D-modules representing regular differential equations

↑ monodromy data, representations of $\pi_1(X)$

ANALYSIS ← – – – ALGEBRA – – – → TOPOLOGY

Theorem (M. Kashiwara 1984)

Masaki Kashiwara
(*1947)

There is an equivalence of categories

$$\{\textit{regular holonomic D-modules on } X\} \xrightarrow{\sim} \{\textit{perverse sheaves on } X\}$$

How about *irregular* equations?

Hilbert's (Deligne's, Kashiwara's...) philosophy:

$$\{\text{regular differential equations}\} \longleftrightarrow \{\text{monodromy data}\}$$

Question

Can we generalize this?

$$\{\text{differential equations}\} \quad \longleftrightarrow \quad \{???\}$$

↑ possibly irregular differential equations

↑ **We need more data about the behaviour of the solutions than just the monodromy!**

How about *irregular* equations?

Important example of an irregular equation:

$$f''(z) = z \cdot f(z)$$

Airy differential equation

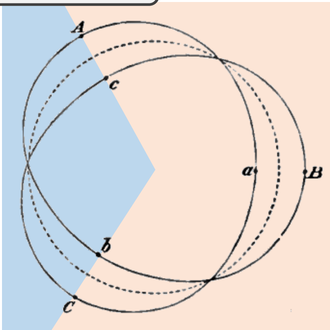
solutions: $f_1(z) = \mathrm{Ai}(z), \qquad f_2(z)$

Stokes matrix

$$\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$$

$$f_1(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{\frac{3}{2}}} + i \cdot \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3} z^{\frac{3}{2}}}$$

$$f_2(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3} z^{\frac{3}{2}}}$$



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Irregular Riemann–Hilbert correspondences

Hilbert's (Deligne's, Kashiwara's...) philosophy:

$$\{\text{regular differential equations}\} \longleftrightarrow \{\text{monodromy data}\}$$

Idea for a generalization

There should be a correspondence

$$\{\text{differential equations}\} \longleftrightarrow \{\text{monodromy data} + \text{Stokes data}\}$$

With this idea, Deligne's and Kashiwara's results could be generalized.

(Deligne–Malgrange 1979, D'Agnolo–Kashiwara 2016)

monodromy data: record changes after a full loop

Stokes data: record asymptotic changes from sector to sector

Stokes in love

PEMBROKE COLLEGE,
May 11*th*, 1857.

......If we are married at the time we are at present thinking of, and go to Switzerland as we talked of, I think I will bring a couple of quartz prisms, a quartz lens, and a piece of uranium glass with me, to observe the spectrum on top of the Rigi or Faulhorn.

from: *Memoir and Scientific Correspondence of the Late Sir George Gabriel Stokes, Bart.*, Selected and Arranged by Joseph Larmor, vol. I, Cambridge University Press (1907), p. 66