D-modules of pure Gaussian type and topological Laplace transform

Andreas Hohl, M.Sc. (hons)

Supervisor: Prof. Dr. Marco Hien

Institute of Mathematics, University of Augsburg, Germany

Universität Augsburg University

Abstract

We consider differential systems of pure Gaussian type. They are D-modules on the complex projective line with an irregular singularity at infinity, and as such are subject to the Stokes phenomenon. The aim is to describe the Stokes matrices attached to the Laplace transform of such a system in terms of the Stokes matrices attached to the original system. We use the theory of enhanced (ind-)sheaves and the Riemann-Hilbert correspondence of D'Agnolo–Kashiwara (see [1]) in order to formulate the problem in a purely topological way. Under an assumption on the parameters, we prove an explicit result about the topological Laplace transform of enhanced sheaves of pure Gaussian type. This also gives an alternative proof of a result of Sabbah (see [3]), which has originally been stated in the context of Stokes-filtered local systems.

D-modules of pure Gaussian type

$X = \mathbb{P}^1$ (analytic) complex projective line

$\mathcal{D}_X$ sheaf of holomorphic differential operators on X

Let $C \subset \mathbb{C} \smallsetminus \{0\}$ be a finite set of parameters.

A holonomic $\mathcal{D}_X$-module $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ is of *pure Gaussian type C* if

- $\mathcal{M}(*\infty) \cong \mathcal{M}$,
- $\mathrm{singsupp}(\mathcal{M}) = \{\infty\}$,
- the Levelt–Turrittin decomposition of the formal completion at ∞ is given by

$$\widehat{\mathcal{M}|_\infty} \simeq \bigoplus_{c\in C} (\mathcal{E}^{-\frac{c}{2}z^2} \overset{\mathrm{D}}{\otimes} R_c)\widehat{|_\infty}$$

with regular holonomic $\mathcal{D}_X$-modules R_c and the affine coordinate $z \in \mathbb{A}^1 \subset X$. Set $r_c := \operatorname{rank} R_c$.

Enhanced Riemann–Hilbert

A. D'Agnolo and M. Kashiwara [1] introduced the category of enhanced ind-sheaves $\mathsf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ with the formalism of six operations. An important role is played by the stabilization functor $\mathbb{C}_X^{\mathsf{E}} \overset{+}{\otimes} (\bullet)$.

Riemann–Hilbert correspondence for holonomic D-modules (see [1]):
The functor of enhanced solutions

$$\mathcal{S}ol_X^{\mathsf{E}} \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathsf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$$

is fully faithful.

In particular, the topological object $\mathcal{S}ol_X^{\mathsf{E}}(\mathcal{M})$ encodes all of $\mathcal{M}$.

Stokes phenomenon in the pure Gaussian case

$\mathcal{M}$ of pure Gaussian type $\rightsquigarrow$ enhanced solutions can be written in the form

$$\mathcal{S}ol_X^{\mathsf{E}}(\mathcal{M}) \simeq \mathbb{C}_X^{\mathsf{E}} \overset{+}{\otimes} \mathcal{F},$$

where $\mathcal{F} \in \mathrm{Mod}(\mathbb{C}_{\mathbb{A}^1\times\mathbb{R}})$ is an enhanced sheaf on $\mathbb{A}^1$ (a usual sheaf on $\mathbb{A}^1 \times \mathbb{R}$).

Stokes lines for a pair $c, d \in C$: rays (in the affine variable z) where

$$\mathrm{Re}\left(-\frac{c}{2}z^2\right) = \mathrm{Re}\left(-\frac{d}{2}z^2\right).$$

Local decomposition of $\mathcal{F}$ into a direct sum of exponentials:
Cover the plane by four sectors, each containing exactly one Stokes line for every pair $c, d \in C$.

exponential enhanced sheaf
$\mathsf{E}^{\varphi} := \mathbb{C}_{\{(z,t)\in\mathbb{A}^1\times\mathbb{R}\,|\,t+\mathrm{Re}\,\varphi(z)\geq 0\}}$

There are isomorphisms

$$\alpha_j \colon \mathcal{F}|_{S_j\times\mathbb{R}} \xrightarrow{\simeq} \bigoplus_{c\in C} (\mathsf{E}^{-\frac{c}{2}z^2})^{r_c}|_{S_j\times\mathbb{R}}$$

and transition maps $\alpha_{j+1}|_{S_{j,j+1}\times\mathbb{R}} \circ \alpha_j^{-1}|_{S_{j,j+1}\times\mathbb{R}}$ represented by Stokes matrices σ_j.

Properties of Stokes matrices:

Choice of a generic direction θ_0 induces an ordering on C
$\rightsquigarrow$ Stokes matrices are upper/lower triangular block matrices.

The monodromy is trivial: $\sigma_4\sigma_3\sigma_2\sigma_1 = \mathrm{id}$.

Laplace transform

Diagrams of projections:

$$\begin{tikzcd} & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \underset{\text{affine variable } z}{\mathbb{P}^1} & & \underset{\text{affine variable } w}{\mathbb{P}^1} \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{R} \arrow[dl, "\widetilde{p}"'] \arrow[dr, "\widetilde{q}"] & \\ \mathbb{A}^1 \times \mathbb{R} & & \mathbb{A}^1 \times \mathbb{R} \end{tikzcd}$$

Laplace transform for $\mathcal{D}_X$-modules

$\mathcal{M} \longmapsto {}^{\mathsf{L}}\mathcal{M} := \mathrm{D}q_*(\mathcal{E}^{-zw} \otimes^{\mathrm{D}} \mathrm{D}p^{-1}\mathcal{M})$

Topological Laplace transform for enhanced sheaves (cf. [2])

$\mathcal{F} \longmapsto {}^{\mathsf{L}}\mathcal{F} := \mathrm{R}\widetilde{q}_!(\mathsf{E}^{-zw} \overset{+}{\otimes} \widetilde{p}^{-1}\mathcal{F})$

Compatibility: For $\mathcal{M}$ of pure Gaussian type with $\mathcal{S}ol_X^{\mathsf{E}}(\mathcal{M}) \simeq \mathbb{C}_X^{\mathsf{E}} \overset{+}{\otimes} \mathcal{F}$, one has

$$\mathcal{S}ol_X^{\mathsf{E}}({}^{\mathsf{L}}\mathcal{M}) \simeq \mathbb{C}_X^{\mathsf{E}} \overset{+}{\otimes} {}^{\mathsf{L}}\mathcal{F}.$$

Result for aligned parameters

Let $C = \{c_1, \ldots, c_n\}$, where all the c_j have the same argument $\arg C$ and are ordered by increasing absolute values. In this case, there are four Stokes lines.

Theorem

Set $\theta_0 := -\frac{1}{2}\arg C$. Consider the four sectors as shown in Figure 1.
Assume that $\mathcal{F} \in \mathrm{Mod}(\mathbb{C}_{\mathbb{A}^1\times\mathbb{R}})$ decomposes as a direct sum of exponentials on sectors

$$\mathcal{F}|_{S_j\times\mathbb{R}} \simeq \bigoplus_{j=1}^{n} (\mathsf{E}^{-\frac{c_j}{2}z^2})^{r_j}|_{S_j\times\mathbb{R}} \qquad \text{for } r_j \in \mathbb{Z}_{>0}$$

with gluing matrices σ_j.

Then the object ${}^{\mathsf{L}}\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{A}^1\times\mathbb{R}})$

- is concentrated in degree 0, i.e. $\mathcal{F} \in \mathrm{Mod}(\mathbb{C}_{\mathbb{A}^1\times\mathbb{R}})$,
- decomposes on the sectors defined by $\widehat{\theta}_0 := \pi - \theta_0$ (see Figure 2)

$${}^{\mathsf{L}}\mathcal{F}|_{\widehat{S}_j\times\mathbb{R}} \simeq \bigoplus_{j=1}^{n} (\mathsf{E}^{\frac{1}{2c_j}w^2})^{r_j}|_{\widehat{S}_j\times\mathbb{R}}$$

and the gluing matrices are given by $\widehat{\sigma}_j = \sigma_j$.

The proof only involves computations in the category of sheaves of vector spaces. Gluing can be modelled by short exact sequences, to which we apply the topological Laplace transformation functor in order to obtain distinguished triangles describing ${}^{\mathsf{L}}\mathcal{F}$.

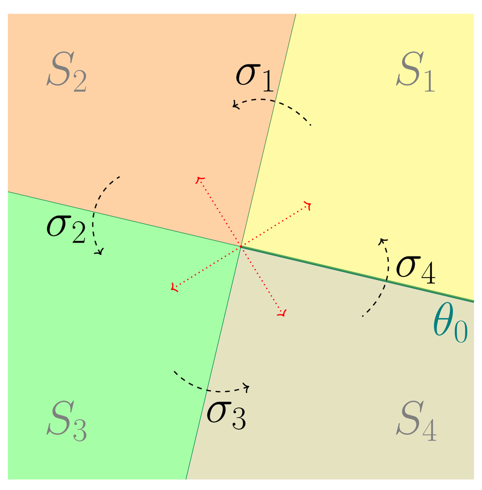

Figure 1: Original system

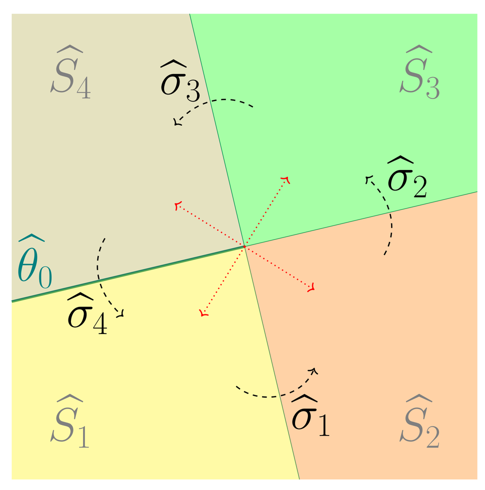

Figure 2: Laplace transform

$\mathbb{A}^1$ covered by four closed sectors bounded by half-lines from the origin with directions $\theta_0 + k\frac{\pi}{2}$ (resp. $\widehat{\theta}_0 + k\frac{\pi}{2}$). The red arrows indicate the Stokes lines.

Corollary (cf. Sabbah [3])

Let $\mathcal{M}$ be a $\mathcal{D}_X$-module of pure Gaussian type C and $(\sigma_j)_{j\in\mathbb{Z}/4\mathbb{Z}}$ its Stokes matrices with respect to the generic direction $\theta_0 = -\frac{1}{2}\arg C$. Then the Laplace transform ${}^{\mathsf{L}}\mathcal{M}$ is of pure Gaussian type $\widehat{C} = -\frac{1}{C}$ and its Stokes matrices $(\widehat{\sigma}_j)_{j\in\mathbb{Z}/4\mathbb{Z}}$ with respect to the generic direction $\widehat{\theta}_0 = \pi - \theta_0$ are equal to $(\sigma_j)_{j\in\mathbb{Z}/4\mathbb{Z}}$.

This corollary corresponds to results of Sabbah [3], who formulated it in the language of Stokes data associated with Stokes-filtered local systems.

References

[1] A. D'Agnolo and M. Kashiwara. *Riemann-Hilbert correspondence for holonomic D-modules.* Publ. Math. Inst. Hautes Études Sci. **123** (2016), no. 1, 69–197.

[2] M. Kashiwara and P. Schapira. *Irregular holonomic kernels and Laplace transform.* Selecta Math. **22** (2016), no. 1, 55–109.

[3] C. Sabbah. *Differential systems of pure Gaussian type.* Izvestiya: Math. **80** (2016), no. 1, 189–220.