Moderate and Rapid Decay Cycles using Enhanced Ind-Sheaves

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Abstract

We define moderate growth and rapid decay objects associated to any enhanced ind-sheaf on a complex manifold X of arbitrary dimension and a holomorphic function $f\colon X \to \mathbb{C}$. We show that these objects recover in particular—via the irregular Riemann–Hilbert correspondence of D'Agnolo–Kashiwara—the moderate growth and rapid decay De Rham complexes of a holonomic $\mathcal{D}_X$-module. This also enables us to deduce well-known duality pairings by topological means. Moreover, our new objects admit natural distinguished triangles, generalizing those known from the classical theory of nearby and vanishing cycles for constructible sheaves.

Moderate growth and rapid decay

Consider a complex manifold X and a holomorphic function $f\colon X \to \mathbb{C}$

$\rightsquigarrow$ One can define $\varpi\colon \widetilde{X}_f \longrightarrow X$, the ***real blow-up space of** X **along*** $f^{-1}(0)$ with sheaves $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f}$ and $\mathcal{A}^{\mathrm{rd}}_{\widetilde{X}_f}$ of functions being holomorphic in the interior and having *moderate growth* and *rapid decay* at $\partial\widetilde{X}_f$, respectively.

Moderate and rapid decay De Rham complexes:

When $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, one defines

$$\mathcal{DR}^{\star}_{\widetilde{X}_f}\mathcal{M} := \varpi^{-1}\Omega_X \otimes_{\varpi^{-1}\mathcal{D}_X} (\mathcal{A}^{\star}_{\widetilde{X}_f} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{M}) \qquad \text{for } \star \in \{\mathrm{mod}, \mathrm{rd}\}.$$

Why do we care about these objects?

These complexes are essential in the study of systems with ***irregular singularities***—the growth behaviour of solutions plays an important role in the classification of the latter. For instance, it is the main ingredient in the definition of the ***Stokes filtrations*** associated to holonomic systems on curves.

Enhanced nearby and vanishing cycles

The category of ***enhanced ind-sheaves*** $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ has been introduced by D'Agnolo–Kashiwara.

Enhanced Riemann–Hilbert for holonomic D-modules (D'Agnolo–Kashiwara [1]):

The enhanced De Rham functor

$$\mathcal{DR}^{\mathrm{E}}_X\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \longrightarrow \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$$

is a fully faithful functor from the category of holonomic $\mathcal{D}_X$-modules to the category of enhanced ind-sheaves.

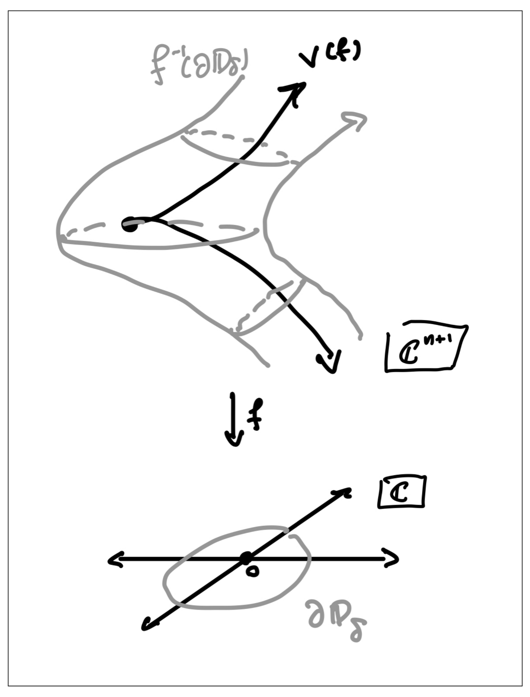
How should I think of an enhanced ind-sheaf?

An enhanced ind-sheaf is a generalized version of a sheaf of vector spaces. In the context of general holonomic $\mathcal{D}_X$-modules, enhanced ind-sheaves play the role that more classical objects like local systems, constructible sheaves and perverse sheaves played in the context of systems with regular singularities: They all serve as ***topological counterparts*** of these analytic objects. In particular, all information about a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is encoded in the object $\mathcal{DR}^{\mathrm{E}}_X(\mathcal{M})$.

We consider a complex manifold X with a holomorphic function $f\colon X \to \mathbb{C}$, and the maps

$$\begin{tikzcd} \partial\widetilde{X}_f \arrow[r, hook, "i"] & \widetilde{X}_f \arrow[d, "\varpi"'] & X^*_\infty \arrow[l, hook', "j^f"'] \arrow[dl, hook, "j"] \\ & X & \end{tikzcd}$$

Here, X^*_∞ is a "bordered space" attached to the interior $X^* := X \setminus f^{-1}(0) = \widetilde{X}_f \setminus \partial\widetilde{X}_f$, i.e. it is a version of X^* that "remembers information about its boundary".



Visualizing the real blow-up $\widetilde{X}_f$:

The boundary of the Milnor tube around $V(f) = f^{-1}(0)$, which is the preimage $f^{-1}(\partial\mathbb{D}_\delta)$, is equal to the boundary $\partial\widetilde{X}_f$ of the real blow-up of X along f. Here, $\mathbb{D}_\delta$ denotes a small disk around the origin in $\mathbb{C}$.

Moderate growth and rapid decay for enhanced ind-sheaves

Motivated by a recent work of D'Agnolo–Kashiwara [2] in dimension one, we define for an object $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ the following complexes of sheaves in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}_f})$:

$$K^{\mathrm{mod}\, f} := \mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}j^f_* \mathrm{E}j^{-1}K\big) \qquad \text{and} \qquad K^{\mathrm{rd}\, f} := \mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}j^f_{!!} \mathrm{E}j^{-1}K\big).$$

If $K = \mathcal{DR}^{\mathrm{E}}_X(\mathcal{M})$ is the enhanced De Rham complex of a holonomic $\mathcal{D}_X$-module, the first functor recovers its moderate growth De Rham complex. In other words, there is a commutative triangle

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[rr, "\mathcal{DR}^{\mathrm{E}}_X"] \arrow[dr, "\mathcal{DR}^{\mathrm{mod}}_{\widetilde{X}_f}"'] & & \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X) \arrow[dl, "(-)^{\mathrm{mod}\, f}"] \\ & \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}_f}) & \end{tikzcd}$$

Duality statements

- If K is an $\mathbb{R}$-constructible enhanced ind-sheaf, it follows directly from the definition that

$$\mathrm{D}_{\widetilde{X}_f}\big(K^{\mathrm{mod}\, f}\big) \simeq \big(\mathrm{D}^{\mathrm{E}}_X K\big)^{\mathrm{rd}\, f}.$$

- **It is expected that** results of Kashiwara–Schapira [3] on the duality between tempered and Whitney holomorphic functions enable us to show that

$$\mathrm{D}_{\widetilde{X}_f}\mathcal{DR}^{\mathrm{mod}}_{\widetilde{X}_f}(\mathcal{M}) \simeq \mathcal{DR}^{\mathrm{rd}}_{\widetilde{X}_f}(\mathbb{D}_X\mathcal{M}).$$

This duality of De Rham complexes would then also prove a recent conjecture of C. Sabbah in [4].

The duality of moderate growth and rapid decay then gives a commutative diagram

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[rr, "\mathcal{DR}^{\mathrm{E}}_X"] \arrow[dr, "\mathcal{DR}^{\mathrm{rd}}_{\widetilde{X}_f}"'] & & \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X) \arrow[dl, "(-)^{\mathrm{rd}\, f}"] \\ & \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}_f}) & \end{tikzcd}$$

Natural distinguished triangles associated to enhanced cycles

- By appropriately restricting the functors $(-)^{\mathrm{mod}\, f}$ and $(-)^{\mathrm{rd}\, f}$ to the boundary $\partial\widetilde{X}_f$, we obtain functors

$$\psi^{\star}_f\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\partial\widetilde{X}_f})$$

(for $\star = \leq 0, < 0, \geq 0, > 0, *$, corresponding to $\mathrm{mod}, \mathrm{rd}, > \mathrm{rd}, > \mathrm{mod}, *$ in [4], respectively).

- These functors fit into two distinguished triangles in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\partial\widetilde{X}_f})$

$$\psi^{<0}_f K \longrightarrow \psi^{*}_f K \xrightarrow{\;c\;} \psi^{\geq 0}_f K \xrightarrow{+1}$$

$$\psi^{\leq 0}_f K \xrightarrow{\;v\;} \psi^{*}_f K \longrightarrow \psi^{> 0}_f K \xrightarrow{+1}$$

that are functorial in $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$. We moreover show that these distinguished triangles are compatible with proper base changes between complex manifolds.

If $K = e_X(F)$ is the enhanced ind-sheaf associated to some $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$, then the above distinguished triangles are functorially isomorphic to the following **canonical** and **variation** distinguished triangles associated to the nearby and vanishing cycle functors (denoted $\psi_f[-1]F$ and $\varphi_f[-1]F$, resp.):

$$i_f^{-1}[-1]F \longrightarrow \psi_f[-1]F \xrightarrow{\mathrm{can}} \varphi_f[-1]F \xrightarrow{+1}$$

$$\varphi_f[-1]F \xrightarrow{\mathrm{var}} \psi_f[-1]F \longrightarrow i_f^{!}[1]F \xrightarrow{+1}$$

For this reason, we refer to the functors $\psi^{\star}_f$ as ***enhancements of the nearby and vanishing cycles functors***.

Duality pairings

We can reformulate the above duality as a perfect pairing

$$K^{\mathrm{mod}\, f} \otimes_{\mathbb{C}} \big(\mathrm{D}^{\mathrm{E}}_X K\big)^{\mathrm{rd}\, f} \longrightarrow \omega_{\widetilde{X}_f}$$

for any $\mathbb{R}$-constructible $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ and $f : X \to \mathbb{C}$ holomorphic.
This also yields a local duality pairing

$$\mathcal{DR}^{\mathrm{mod}}_{\widetilde{X}_f}(\mathcal{M}) \otimes_{\mathbb{C}} \mathcal{DR}^{\mathrm{rd}}_{\widetilde{X}_f}(\mathbb{D}_X\mathcal{M}) \longrightarrow \mathcal{DR}^{\mathrm{rd}}_{\widetilde{X}_f}(\mathcal{O}_X, d) \simeq j^f_!\mathbb{C}_{X^*}[2n],$$

where $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, and d is the usual exterior (complex) derivative on the sheaf of holomorphic functions $\mathcal{O}_X$, and n is the (complex) dimension of X.

Bloch–Esnault and Hien's rapid decay homology

In particular, if (E, ∇) denotes a flat (algebraic) connection on a smooth quasi-projective variety U over $\mathbb{C}$, and $(E^\vee, \nabla^\vee)$ denotes the dual connection on U, we recover the well-known perfect pairing of finite-dimensional $\mathbb{C}$-vector spaces

$$\mathrm{H}^m_{\mathrm{dR}}\big(U; (E, \nabla)\big) \otimes_{\mathbb{C}} \mathrm{H}^{\mathrm{rd}}_m\big(U; (E^\vee, \nabla^\vee)\big) \to \mathbb{C}$$

where H_{dR} denotes algebraic De Rham cohomology, and H^{rd} denotes the rapid decay homology of Bloch–Esnault [5] and Hien [6].

References

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